Hamiltonization of nonholonomic systems

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Abstract

We consider some issues of the representation in the Hamiltonian form of two problems of nonholonomic mechanics, namely, the Chaplygin's ball problem and the Veselova problem. We show that these systems can be written as generalized Chaplygin systems and can be integrated by the method of reducing multiplier. We also indicate the algebraic form of the Poisson brackets of these systems (after the time substitution). Generalizations of the problems are considered and new realizations of nonholonomic constraints are presented. Some nonholonomic systems with an invariant measure and a sufficient number of first integrals are indicated, for which the question of the representation in the Hamiltonian form is still open, even after the time substitution.

First, we consider certain general results related to the method of integration of nonholonomic systems, which S. A. Chaplygin [3] called the method of reducing multiplier. We generalize this method so it can apply to a broader class of systems, so-called generalized Chaplygin systems. In the remaining sections, we apply these results to find explicitly the Poisson structures and the isomorphisms with other integrable Hamiltonian systems.

1. Generalized Chaplygin systems

Consider a mechanical system with two degrees of freedom, such that the equations of motion can be written in the form:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = \dot{q}_2 S, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) - \frac{\partial L}{\partial q_2} = -\dot{q}_1 S, \qquad S = a_1(q)\dot{q}_1 + a_2(q)\dot{q}_2 + b(q), \tag{1}$$

where L is a function of coordinates and velocities; this function will be also referred to as the Lagrangian of the system.

For a special form of S with $b(q) = 0$ we obtain an ordinary Chaplygin system [3]. S. A. Chaplygin showed that the equations of so-called Chaplygin sleigh can be reduced to

the form (1) (with $b(q) = 0$); the system (1) can be integrated by applying the below-stated method of reducing multiplier and the solution of the Hamilton–Jacobi equation. In [1] the authors show that the Veselova system is a Chaplygin system. The Veselova system describes the rotation of a rigid body about a fixed point under a nonintegrable constraint: the projection of its angular velocity onto the fixed axis is zero. We show below that the system of equations for Chaplygin's ball on a plane [6] can also be reduced to the form (1) (but with $b(q) \neq 0$).

We will call the system (1) *a generalized Chaplygin system* (it should not be confused with that from [22, 30], where a different generalization of Chaplygin systems was offered!).

Chaplygin showed that in the case $b(q) = 0$ [3], the equations preserve their form under time substitutions

$$N(q)\,dt = d\tau$$

if N does not depend on the velocities. Let us show that this also holds for equations (1).

Denoting the differentiation by $q_i' = \dfrac{dq_i}{d\tau}$ we find

$$\dot{q}_i = Nq_i', \quad \frac{\partial L}{\partial \dot{q}_i} = \frac{1}{N}\frac{\partial \bar{L}}{\partial q_i'}, \quad \frac{\partial L}{\partial q_i} = \frac{\partial \bar{L}}{\partial q_i} - \frac{1}{N}\frac{\partial N}{\partial q_i}\sum_{k=1}^{2} q_k' \frac{\partial \bar{L}}{\partial q_k'},$$

where $\bar{L}(q, q') = L(q, Nq')$.

Substituting into (1) gives

$$\begin{gathered} \frac{d}{d\tau}\left(\frac{\partial \bar{L}}{\partial q_q'}\right) - \frac{\partial \bar{L}}{\partial q_1} = q_2'\bar{S}, \quad \frac{d}{d\tau}\left(\frac{\partial \bar{L}}{\partial q_2'}\right) - \frac{\partial \bar{L}}{\partial q_2} = -q_1'\bar{S}, \\ \bar{S} = NS + \frac{1}{N}\left(\frac{\partial N}{\partial q_2}\frac{\partial \bar{L}}{\partial q_1'} - \frac{\partial N}{\partial q_1}\frac{\partial \bar{L}}{\partial q_2'}\right). \end{gathered} \tag{2}$$

It is known for an ordinary Chaplygin system [3] that if there is an invariant measure with density depending only on the coordinates, one can choose $N(q)$ for which $\bar{S} = 0$; hence, in terms of the new time τ, the system can be written in the classical Hamiltonian form. Let us give a generalization of this result for the case of generalized Chaplygin systems of the form (1), under the assumption that the Lagrangian is a quadratic function of the velocities $\dot{q}_i$ (not necessarily homogeneous).

Theorem 1. *Let* $\det\left\|\dfrac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right\| \neq 0$, *and let the system* (1) *admit an invariant measure with density depending only on the coordinates; then there is a time substitution* $N(q)\,dt = d\tau$ *such that*

1) *the function* $\bar{S}$, *defined by* (2), *depends only on the coordinates*: $\bar{S} = \bar{S}(q)$,

2) *in terms of the new time, the equations of motion can be written in Hamiltonian form:*

$$\frac{dq_i}{d\tau} = \{q_i, \bar{H}\}, \quad \frac{dp_i}{d\tau} = \{p_i, \bar{H}\},$$

where

$$p_i = \frac{\partial \bar{L}}{\partial q_i'}, \quad \bar{H} = \sum_{k=1}^{2} p_k q_k' - \bar{L}\Big|_{q_i' \to p_i},$$

and the Poisson bracket is given by

$$\{q_i, p_j\} = \delta_{ij}, \quad \{p_1, p_2\} = \bar{S}(q), \quad \{q_1, q_2\} = 0. \tag{3}$$

Proof

Let us apply the Legendre transformation to the initial system (1):

$$P_i = \frac{\partial L}{\partial \dot{q}_i}, \quad H = \sum_i P_i \dot{q}_i - L\Big|_{\dot{q}_i \to P_i};$$

here,

$$\begin{gathered}\dot{q}_i = \frac{\partial H}{\partial P_i}, \quad \dot{P}_1 = -\frac{\partial H}{\partial q_1} + \frac{\partial H}{\partial P_2} S, \quad \dot{P}_2 = -\frac{\partial H}{\partial q_2} - \frac{\partial H}{\partial P_1} S, \\ S = a_1(q)\dot{q}_1 + a_2(q)\dot{q}_2 + b(q) = A_1(q)P_1 + A_2(q)P_2 + B(q).\end{gathered} \tag{4}$$

Using the Liouville equation for the density of the invariant measure $\rho(q)\, dP_1\, dP_2\, dq_1\, dq_2$ of the system (4), we find

$$\dot{q}_1 \left(\frac{1}{\rho} \frac{\partial \rho}{\partial q_1} - A_2(q) \right) + \dot{q}_2 \left(\frac{1}{\rho} \frac{\partial \rho}{\partial q_2} + A_1(q) \right) = 0;$$

since ρ depends only on the coordinates, each of the brackets should become zero separately:

$$\frac{1}{\rho} \frac{\partial \rho}{\partial q_1} - A_2(q) = 0, \quad \frac{1}{\rho} \frac{\partial \rho}{\partial q_2} + A_1(q) = 0.$$

Now we write equation (2) for $\bar{S}$, taking into account the relation $\dfrac{1}{N} \dfrac{\partial \bar{L}}{\partial q_i'} = P_i$:

$$\bar{S} = \left(N A_1(q) + \frac{\partial N}{\partial q_2} \right) P_1 + \left(N A_2(q) - \frac{\partial N}{\partial q_1} \right) P_2 + B(q).$$

Thus, if we choose $N(q) = \rho(q)$, then $\bar{S} = B(q)$, and the first statement of the theorem is proved.

The second statement can be proved with a straightforward verification of the equations and of the Jacobi identity.

Remark 1. The time substitution, or the reducing multiplier, for multidimensional systems is closely related to the invariant measure, but nevertheless, as shown in [28], can differ from it.

Note also that reduction to the Hamiltonian form is useful for application of the methods of the perturbation theory, introduction of action-angle variables, analysis of integrability and non-integrability, etc.

The Hamiltonian systems with the bracket (3) are used for description of systems with a generalized potential (for example, motion of charged particles in a magnetic field) or systems with gyroscopic forces [25]. In this case, the closed 2-form $\bar{S}(q)\,dq_1 \wedge dq_2$ is called the 2-form of gyroscopic forces. Locally, Ω can be represented as the exact differential $\Omega = d\omega$, $\omega = W_1(q)dq_1 + W_2(q)dq_2$ and the equations of motions (2) can be written in the form of the Lagrange–Euler equations

$$\frac{d}{d\tau}\left(\frac{\partial L_W}{\partial q_i'}\right) - \frac{\partial L_W}{\partial q_i} = 0,$$
$$L_W = \bar{L} + W(q, q'), \quad W(q, q') = W_1(q)q_1' + W_2(q)q_2',$$

here the Poisson brackets for the new momenta $\widetilde{p}_i = p_i + W_i(q)$ and coordinates q_i are canonical (i. e. $\{q_i, \tilde{p}_j\} = \delta_{ij}$).

If the manifold $\mathcal{M}$, on which the coordinates q_1, q_2 are defined is compact, then the criterion for Ω to be exact is

$$\int\limits_{\mathcal{M}} \Omega = 0.$$

Thus, if $\int \Omega \neq 0$ (i. e. the 2-form is not exact) then the generalized potential W and the corresponding Lagrangian and Hamiltonian functions have singularities (so-called monopoles) [23]. In this case, sometimes it is said that the global representation of the equations of motion in the (canonical) Hamiltonian form is impossible.

2. Veselova system

The Veselova system describes the motion of a rigid body with a fixed point subject to nonholonomic constraint of the form $(\boldsymbol{\omega}, \boldsymbol{\gamma}) = 0$, where $\boldsymbol{\omega}$ and $\boldsymbol{\gamma}$ are the body's angular velocity vector and the unit vector of the space-fixed axis in the frame of reference fixed to the body. Thus, for the Veselova constraint, the projection of the angular velocity onto a space-fixed axis is zero. This constraint is reciprocal to the Suslov constraint [24], for which the projection of the angular velocity onto a body-fixed axis is zero.

In the moving axes fixed to the body, the equations of motion can be written as follows [5, 7]:

$$\mathbf{I}\dot{\boldsymbol{\omega}} = \mathbf{I}\boldsymbol{\omega} \times \boldsymbol{\omega} + \mu\boldsymbol{\gamma} + \boldsymbol{\gamma} \times \frac{\partial U}{\partial \boldsymbol{\gamma}}, \quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\omega}, \tag{5}$$

where μ is an undetermined multiplier, $\mathbf{I} = \mathrm{diag}\,(I_1, I_2, I_3)$ is the tensor of inertia, $U(\boldsymbol{\gamma})$ is the potential energy. The undetermined multiplier μ can be found by differentiating the constraint:

$$\mu = \frac{\left(\mathbf{I}\boldsymbol{\omega} \times \boldsymbol{\omega} + \boldsymbol{\gamma} \times \frac{\partial U}{\partial \boldsymbol{\gamma}}, \mathbf{I}^{-1}\boldsymbol{\gamma}\right)}{(\boldsymbol{\gamma}, \mathbf{I}^{-1}\boldsymbol{\gamma})}. \tag{6}$$

In the general case the equations (5) admit the integral of energy and the geometric integral

$$H = \frac{1}{2}(\mathbf{I}\boldsymbol{\omega}, \boldsymbol{\omega}) + U(\boldsymbol{\gamma}), \quad \boldsymbol{\gamma}^2 = 1, \tag{7}$$

as well as the invariant measure $\rho_\omega \, d^3\boldsymbol{\omega} \, d^3\boldsymbol{\gamma}$ with density

$$\rho_\omega = \sqrt{(\boldsymbol{\gamma}, \mathbf{I}^{-1}\boldsymbol{\gamma})}. \tag{8}$$

When $U = 0$, there is an additional integral

$$F = (\mathbf{I}\boldsymbol{\omega}, \mathbf{I}\boldsymbol{\omega}) - (\mathbf{I}\boldsymbol{\omega}, \boldsymbol{\gamma})^2 = |\mathbf{I}\boldsymbol{\omega} \times \boldsymbol{\gamma}|^2, \tag{9}$$

and hence, the system is integrable according to the Euler–Jacobi theorem [5].

Remark 2. The integral (9) is generalized when the Brun potential is added [5, 7]. Some other integrable potentials are given in [1, 13].

Remark 3. The system (5), (6) with the constraint $(\boldsymbol{\omega}, \boldsymbol{\gamma}) = 0$ was rediscovered in paper [15] almost ten years after [5, 7]. In [15], an explicit integration was performed using sphero-conical coordinates.

Remark 4. The Veselova system and the nonholonomic systems (considered below) describing the rolling motion of bodies belong to the class of so-called LR- and $L + R$-systems on Lie groups [1, 7]. Several results on the existence of invariant measure for such systems are known. We do not consider here these general results, especially useful for multidimensional generalizations. Note also that the general methods of reduction of nonholonomic systems were examined in many papers, see for example [19].

Remark 5. Generalization of the Veselova constraint $(\boldsymbol{\omega}, \boldsymbol{\gamma}) = d \neq 0$ was considered in [9]. Using Chaplygin's method of integration for a dynamically asymmetric ball with non-zero constant of areas [6], the author presented an explicit integration of the equations.

It was shown in [1] that the Veselova system is the Chaplygin system (1) with $b(q) = 0$ and, therefore, upon the time substitution $N\,dt = d\tau$, it can be written in the Hamiltonian form, where the reducing multiplier is $N = \rho_\omega^{-1}$. Let us show this explicitly using the local coordinates (namely, the Euler angles θ, φ, ψ) and then apply the obtained canonical Poisson structure of the cotangent bundle of the sphere T^*S^2 to construct an algebraic Poisson bracket of redundant variables $\boldsymbol{\omega}$, $\boldsymbol{\gamma}$. With such an algebraization of the Poisson structure, one can naturally establish an isomorphism with the Neumann system describing the dynamics of a point on a sphere in a quadratic potential. This isomorphism was straightforwardly established

in [5, 7]. Later, we will see that this analogy can be directly extended to the Chaplygin ball and the general Clebsch system (which includes the Neumann system as a particular case).

In terms of the Euler angles, the body's angular velocity $\boldsymbol{\omega}$ and the unit vector $\boldsymbol{\gamma}$ are given by

$$\boldsymbol{\omega} = (\dot\psi\sin\theta\sin\varphi + \dot\theta\cos\varphi, \dot\psi\sin\theta\cos\varphi - \dot\theta\sin\varphi, \dot\psi\cos\theta + \dot\varphi), \quad \boldsymbol{\gamma} = (\sin\theta\sin\varphi, \sin\theta\cos\varphi, \cos\theta). \tag{10}$$

The equation of the constraint is

$$f = (\boldsymbol{\omega}, \boldsymbol{\gamma}) = \dot\psi + \cos\theta\dot\varphi = 0, \tag{11}$$

Eliminating the undetermined Lagrange multiplier from the equations of motion we can write the equation for θ, φ as a Chaplygin system:

$$\begin{gathered}\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right) - \frac{\partial T}{\partial\theta} + \frac{\partial U}{\partial\theta} = \dot\varphi S, \quad \frac{d}{dt}\left(\frac{\partial T}{\partial\dot\varphi}\right) - \frac{\partial T}{\partial\varphi} + \frac{\partial U}{\partial\varphi} = -\dot\theta S, \\ S = \left.\frac{\partial T_0}{\partial\dot\psi}\right|_{\dot\psi=-\cos\theta\dot\varphi} = \sin^2\theta\left(\dot\theta(I_2 - I_1)\sin\varphi\cos\varphi - \dot\varphi(I_1\cos^2\varphi + I_2\sin^2\varphi + I_3)\right),\end{gathered} \tag{12}$$

where U is the potential energy of the body in an external field, $T_0 = \frac{1}{2}(\boldsymbol{\omega}, \mathbf{I}\boldsymbol{\omega})$ is the kinetic energy without the constraint, while T is the kinetic energy from which $\dot\psi$ is eliminated using the constraint

$$\begin{aligned}T = \left.T_0\right|_{\dot\psi=-\cos\theta\dot\varphi} &= \frac{1}{2}I_1(\dot\theta\cos\varphi - \dot\varphi\sin\varphi\sin\theta\cos\theta)^2 + \\ &+ \frac{1}{2}I_2(\dot\theta\sin\varphi + \dot\varphi\cos\varphi\sin\theta\cos\theta)^2 + \frac{1}{2}I_3\dot\varphi^2\sin^4\theta.\end{aligned} \tag{13}$$

Remark 6. The representation (12) is obtained upon differentiation under the condition (11):

$$\frac{\partial T}{\partial\dot\theta} = \frac{\partial T_0}{\partial\dot\theta}, \quad \frac{\partial T}{\partial\dot\varphi} = \frac{\partial T_0}{\partial\dot\varphi} - \cos\theta\frac{\partial T_0}{\partial\dot\psi}, \quad \frac{\partial T}{\partial\theta} = \frac{\partial T_0}{\partial\theta} + \dot\varphi\sin\theta\frac{\partial T_0}{\partial\dot\psi}, \quad \frac{\partial T}{d\varphi} = \frac{\partial T_0}{\partial\varphi}.$$

Theorem 2 ([1]). *After the time substitution $N\,dt = d\tau$, $N = (\boldsymbol{\gamma}, \mathbf{I}\boldsymbol{\gamma})^{-1/2}$, the equations of motion of the Veselova system take the form of the Euler–Lagrange equations:*

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial\theta'}\right) - \frac{\partial L}{\partial\theta} = 0, \quad \frac{d}{d\tau}\left(\frac{\partial L}{\partial\varphi'}\right) - \frac{\partial L}{\partial\varphi} = 0, \tag{14}$$

where $L = T - U\big|_{\dot\theta=N\theta',\,\dot\varphi=N\varphi'}$ is the Lagrangian function; after time substitution it can be written in the form:

$$L = \frac{1}{2}\frac{(\boldsymbol{\gamma}'\times\boldsymbol{\gamma}, \mathbf{I}(\boldsymbol{\gamma}'\times\boldsymbol{\gamma}))}{(\boldsymbol{\gamma}, \mathbf{I}^{-1}\boldsymbol{\gamma})} - U(\boldsymbol{\gamma}).$$

Proof

The *proof* is based on a simple computation test: after the time substitution, the right-hand side $\bar{S}$ of (14), calculated by virtue of (2), should vanish.

The canonical Hamiltonian form of the equations of motion (14) can be obtained using the Legendre transformation

$$\begin{gathered}p_\theta = \frac{\partial L}{\partial \theta'}, \quad p_\varphi = \frac{\partial L}{\partial \varphi'}, \quad H = p_\theta \theta' + p_\varphi \varphi' - L, \\ \frac{d\theta}{d\tau} = \frac{\partial H}{\partial p_\theta}, \quad \frac{d\varphi}{d\tau} = \frac{\partial H}{\partial p_\varphi}, \quad \frac{dp_\theta}{d\tau} = -\frac{\partial H}{\partial \theta}, \quad \frac{dp_\varphi}{d\tau} = -\frac{\partial H}{\partial \varphi}.\end{gathered} \tag{15}$$

Using the canonical variables of (15) and the time substitution $(\rho_\omega \sqrt{\det \mathbf{I}})^{-1} dt = d\tau$, one can write the equations of motion of the Veselova system in the Hamiltonian form on the (co)algebra of the Poisson brackets $e(3)$:

$$\begin{gathered}\boldsymbol{M} = \rho_\omega \mathbf{I}^{1/2} \boldsymbol{\omega}, \quad \boldsymbol{\Gamma} = \rho_\omega^{-1} \mathbf{I}^{-1/2} \boldsymbol{\gamma}, \\ \frac{d\boldsymbol{M}}{d\tau} = \boldsymbol{M} \times \frac{\partial H}{\partial \boldsymbol{M}} + \boldsymbol{\Gamma} \times \frac{\partial H}{\partial \boldsymbol{\Gamma}}, \quad \frac{d\boldsymbol{\Gamma}}{d\tau} = \boldsymbol{\Gamma} \times \frac{\partial H}{\partial \boldsymbol{M}},\end{gathered} \tag{16}$$

$$H = \frac{1}{2}(\boldsymbol{\Gamma}, \mathbf{I}\boldsymbol{\Gamma})(\boldsymbol{M}, \boldsymbol{M}) + \widetilde{U}(\boldsymbol{\Gamma}), \tag{17}$$

where $\widetilde{U}(\boldsymbol{\Gamma}) = U\left(\rho_\omega \mathbf{I}^{1/2} \boldsymbol{\Gamma}\right)$, and, respectively,

$$\begin{gathered}\boldsymbol{\gamma}^2 = \boldsymbol{\Gamma}^2 = 1, \quad (\boldsymbol{\omega}, \boldsymbol{\gamma}) = (\boldsymbol{M}, \boldsymbol{\gamma}) = 0, \quad \rho_\omega = (\boldsymbol{\gamma}, \mathbf{I}^{-1}\boldsymbol{\gamma})^{1/2} = (\boldsymbol{\Gamma}, \mathbf{I}\boldsymbol{\Gamma})^{-1/2}, \\ \{M_i, M_j\} = \varepsilon_{ijk} M_k, \quad \{M_i, \Gamma_j\} = \varepsilon_{ijk} \Gamma_k, \quad \{\Gamma_i, \Gamma_j\} = 0.\end{gathered}$$

Thus, we have a Hamiltonian system with Poisson brackets corresponding to the algebra $e(3)$, and the four-dimensional symplectic leaf of the structure corresponding to the real motion is given by $\boldsymbol{\gamma}^2 = 1$, $(\boldsymbol{M}, \boldsymbol{\gamma}) = 0$ (for the classical Euler–Poisson equations a similar situation takes place if the constant of areas [23] is zero). Note also that the system (16), (17) determines a certain integrable potential system on a two-dimensional sphere and defines thereby a certain geodesic flow.

The inverse transformation is

$$\boldsymbol{\omega} = (\boldsymbol{\Gamma}, \mathbf{I}\boldsymbol{\Gamma})^{1/2} \mathbf{I}^{-1/2} \boldsymbol{M}, \quad \boldsymbol{\gamma} = (\boldsymbol{\Gamma}, \mathbf{I}\boldsymbol{\Gamma})^{-1/2} \mathbf{I}^{1/2} \boldsymbol{\Gamma}.$$

Therefore, a search for integrable potentials for the Veselova system is now reduced to the well-studied problem of search for integrable cases in a Hamiltonian system on $e(3)$ with Hamiltonian (17). So, if $U = 0$, then the additional integral (9) can be written as

$$F = (\mathbf{I}\boldsymbol{M}, \boldsymbol{M})(\mathbf{I}\boldsymbol{\Gamma}, \boldsymbol{\Gamma}) - (\mathbf{I}\boldsymbol{M}, \boldsymbol{\Gamma})^2.$$

(Note that $\{H, F\} = 0$ only on the level $(\boldsymbol{M}, \boldsymbol{\gamma}) = 0$.)

It was noted in [5, 7] that for $U = 0$ the system (5) is equivalent to the Neumann problem. As we can see, such equivalence is not a result of the natural reduction of the Veselova system to the Hamiltonian form (16), (17) on $e(3)$. It turns out that the isomorphism with the Neumann system is caused by existence of a transformation that does not conserve the Poisson brackets but reduces the vector field to the required form on the level surface $H = \text{const}$.

Indeed, let us consider a Hamiltonian system on $e(3)$ (in the case $(\boldsymbol{M}, \boldsymbol{\gamma}) = 0$) defined by the Hamiltonian

$$H = \alpha \frac{1}{2} \boldsymbol{M}^2 (\boldsymbol{\Gamma}, \mathbf{I}\boldsymbol{\Gamma}) + \beta \frac{1}{2} \left((\boldsymbol{M}, \mathbf{I}\boldsymbol{M})(\boldsymbol{\Gamma}, \mathbf{I}\boldsymbol{\Gamma}) - (\boldsymbol{M}, \mathbf{I}\boldsymbol{\Gamma})^2 \right). \tag{18}$$

It is clear that both terms are the first integrals of the system. The following holds true:

Proposition 1. *On a fixed level* $\dfrac{\boldsymbol{M}^2(\boldsymbol{\Gamma}, \mathbf{I}\boldsymbol{\Gamma})}{\det \mathbf{I}} = c$ *and* $(\boldsymbol{M}, \boldsymbol{\Gamma}) = 0$, *the vector field generated by the Hamiltonian* (18) *is isomorphic to the vector field of the Kirchhoff equations in the Clebsch case with zero value of constant of areas* $(\boldsymbol{L}, \boldsymbol{s}) = 0$; *this field can be written in the form*

$$\begin{aligned} \dot{\boldsymbol{s}} &= k(\alpha \boldsymbol{s} \times \boldsymbol{L} + \beta \boldsymbol{s} \times \mathbf{I}\boldsymbol{L}), \\ \dot{\boldsymbol{L}} &= k\left(\alpha c \boldsymbol{s} \times \mathbf{I}\boldsymbol{s} + \beta(\boldsymbol{L} \times \mathbf{I}\boldsymbol{L} - c(\det \mathbf{I}) \boldsymbol{s} \times \mathbf{I}^{-1}\boldsymbol{s})\right), \end{aligned} \qquad k = -\sqrt{\det \mathbf{I}}. \tag{19}$$

Proof

Let us change the variables

$$\boldsymbol{L} = \mathbf{I}^{-1/2}\boldsymbol{M}, \quad \boldsymbol{s} = (\boldsymbol{\Gamma}, \mathbf{I}\boldsymbol{\Gamma})^{-1/2}\mathbf{I}^{1/2}\boldsymbol{\Gamma},$$

so that the relations $\boldsymbol{s}^2 = \boldsymbol{\Gamma}^2 = 1$, $(\boldsymbol{M}, \boldsymbol{\Gamma}) = (\boldsymbol{s}, \boldsymbol{L}) = 0$ hold. By virtue of linearity we consider the two cases $\alpha = 1$, $\beta = 0$ and $\alpha = 0$, $\beta = 1$ separately. In the first case the equations of motion in terms of the new variables read

$$\begin{aligned} \dot{\boldsymbol{s}} &= -\sqrt{\det \mathbf{I}} \left(\boldsymbol{s} \times \boldsymbol{L} + (\boldsymbol{s}, \boldsymbol{L}) \frac{(\boldsymbol{s} \times \mathbf{I}^{-1}\boldsymbol{s})}{(\boldsymbol{s}, \mathbf{I}^{-1}\boldsymbol{s})} \right), \\ \dot{\boldsymbol{L}} &= -\sqrt{\det \mathbf{I}} \frac{(\mathbf{I}\boldsymbol{L}, \boldsymbol{L})}{\det \mathbf{I} (\boldsymbol{s}, \mathbf{I}^{-1}\boldsymbol{s})} \boldsymbol{s} \times \mathbf{I}\boldsymbol{s}. \end{aligned}$$

Hence, taking into account $(\boldsymbol{s}, \boldsymbol{L}) = 0$, $\dfrac{(\mathbf{I}\boldsymbol{L}, \boldsymbol{L})}{(\boldsymbol{s}, \mathbf{I}^{-1}\boldsymbol{s})} = \boldsymbol{M}^2(\boldsymbol{\Gamma}, \mathbf{I}\boldsymbol{\Gamma}) = c \det \mathbf{I}$, we get the required result.

The case $\alpha = 0$, $\beta = 1$ can be considered analogously.

If we consider c as a constant parameter, then the vector field (19) is generated on $e(3)$ by the following Hamiltonian

$$H = k\alpha \left(\frac{1}{2}\boldsymbol{M}^2 + \frac{c}{2}(\boldsymbol{\Gamma}, \mathbf{I}\boldsymbol{\Gamma}) \right) + k\beta \left(\frac{1}{2}(\boldsymbol{M}, \mathbf{I}\boldsymbol{M}) - \frac{c}{2} \det \mathbf{I} (\boldsymbol{\Gamma}, \mathbf{I}\boldsymbol{\Gamma}) \right).$$

If $\alpha = 1$, $\beta = 0$, we obtain the Hamiltonian of the Neumann case, while at $\alpha = 0$, $\beta = 1$ this is the Hamiltonian of the Brun problem. With arbitrary α, β, this Hamiltonian corresponds to the general Clebsch case in the Kirchhoff equations [12, 23]. Using the representation (16), (17), we easily obtain the following theorem for the Veselova system:

Theorem 3 ([5, 7]). *After time substitution, the vector field of the Veselova problem (with $U = 0$) on the fixed level of the integral of energy $H = h = \mathrm{const}$ becomes isomorphic to the vector field of the Neumann problem.*

3. Chaplygin's ball

Consider the problem of rolling without sliding of a balanced, dynamically asymmetric ball on a horizontal plane in the axisymmetric potential force field (we assume that the geometrical center and the center of mass coincide). We fix a moving frame of reference to the body and write equations of motion in the following form [6]:

$$\begin{gathered}\dot{\boldsymbol{M}} = \boldsymbol{M} \times \boldsymbol{\omega} + \boldsymbol{\gamma} \times \frac{\partial U}{\partial \boldsymbol{\gamma}}, \quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\omega}, \\ \boldsymbol{M} = \mathbf{I}\boldsymbol{\omega} + D\boldsymbol{\gamma} \times (\boldsymbol{\omega} \times \boldsymbol{\gamma}) = \mathbf{I}_Q \boldsymbol{\omega}, \quad D = mR^2,\end{gathered} \tag{20}$$

where $\boldsymbol{\omega}$ is the ball's angular velocity, $\boldsymbol{\gamma}$ is the vertical unit vector in the moving frame of reference, $\mathbf{I} = \operatorname{diag}(I_1, I_2, I_3)$ is the ball's tensor of inertia with respect to its center, m and R are the ball's mass and radius, and $U = U(\boldsymbol{\gamma})$ is the potential of the external axisymmetric field. The vector $\boldsymbol{M}$ is the ball's angular momentum with respect to the point of contact. We present the tensor $\mathbf{I}_Q$ in the form

$$\mathbf{I}_Q = \mathbf{J} - D\boldsymbol{\gamma} \otimes \boldsymbol{\gamma}, \quad \mathbf{J} = \mathbf{I} + D\mathbf{E}.$$

Equations (5) (with an arbitrary potential) admit the integral of energy, the geometric integral and the integral of areas:

$$H = \frac{1}{2}(\boldsymbol{M}, \boldsymbol{\omega}) + U(\boldsymbol{\gamma}), \quad (\boldsymbol{\gamma}, \boldsymbol{\gamma}) = 1, \quad (\boldsymbol{M}, \boldsymbol{\gamma}) = c = \mathrm{const}. \tag{21}$$

They also admit the invariant measure indicated by Chaplygin [6], $\rho_\mu \, d^3\boldsymbol{M} \, d^3\boldsymbol{\gamma}$, with density

$$\rho_\mu = (\det \mathbf{I}_Q)^{-1/2} = \left[\det \mathbf{J}\left(1 - D\left(\boldsymbol{\gamma}, \mathbf{J}^{-1}\boldsymbol{\gamma}\right)\right)\right]^{-1/2}. \tag{22}$$

If there is no external field ($U = 0$), the system (5) has an additional integral

$$F = (\boldsymbol{M}, \boldsymbol{M}), \tag{23}$$

hence, it is integrable according to the Euler–Jacobi theorem [6]. In [6], the solution of (20) was given in terms of hyperelliptic functions.

Remark 7. The integral (23) can be generalized to the cases of the Brun field $U(\boldsymbol{\gamma}) = \frac{k}{2}(\boldsymbol{\gamma}, \mathbf{I}\boldsymbol{\gamma})$ [10] and gyrostat [25]. Other integrable potentials (with the zero constant of areas $(\boldsymbol{M}, \boldsymbol{\gamma}) = 0$) can be found using the representation of the system on algebra $e(3)$, which is given below.

Comment. Although Chaplygin developed the general method of reducing multiplier [3], he did not apply it to the system (20). The paper [10] indicates possible obstructions to the application of this method to the system (20). On the other hand, already in [6] the following question was formulated: if the system (20) can be represented in Hamiltonian form? The problem of Hamiltonization of Chaplygin's ball was formulated more strictly by Kozlov [12] and Duistermaat [11, 17]. In [8], the authors showed numerically that without a time substitution, the equations of motion of the Chaplygin ball are not Hamiltonian because the periods of motion for the orbits lying on the two-dimensional invariant resonance tori are not the same. However, after an appropriate time substitution the system (20) becomes Hamiltonian and Poisson brackets are found explicitly (see Ref. [2]).

Unfortunately, the authors of the review [17] did not succeed in an explicit verifying our result (perhaps, due to certain misprints in [2]). Here, we will prove the result of [2] using another method and then reveal an interesting *isomorphism between the Chaplygin ball and the Clebsch case* in the Kirchhoff equations. Another isomorphism has been indicated in [31].

To describe the ball's rotation let us add to the system (20) the equations for the remaining direction cosines:

$$\dot{\boldsymbol{\alpha}} = \boldsymbol{\alpha} \times \boldsymbol{\omega}, \quad \dot{\boldsymbol{\beta}} = \boldsymbol{\beta} \times \boldsymbol{\omega}.$$

Such a system admits two additional integrals linear in velocities

$$(\boldsymbol{M}, \boldsymbol{\alpha}) = \text{const}, \quad (\boldsymbol{M}, \boldsymbol{\beta}) = \text{const}.$$

Under such an extension, the integral manifolds remain two-dimensional. Hence, the Chaplygin problem with $U = 0$ is degenerate or, as it is sometimes called, superintegrable. From this viewpoint, Chaplygin's ball is a nonholonomic analog of the Euler–Poinsot top, a well-known noncommutative integrable system, and the phase space of this three-degree-of-freedom Hamiltonian system is foliated into two-dimensional tori (not three-dimensional according to the Liouville theorem).

It was shown in [2] that, for an arbitrary potential, after the time substitution and change of variables

$$\rho_\mu \, dt = d\tau, \quad \boldsymbol{L} = \rho_\mu \boldsymbol{M}, \tag{24}$$

the equations of motion (5) take in the Hamiltonian form:

$$\frac{dM_k}{d\tau} = \{H, M_k\}, \quad \frac{d\gamma_k}{d\tau} = \{H, \gamma_k\}$$

with the nonlinear Poisson bracket

$$\{L_i, L_j\} = \varepsilon_{ijk}\left(L_k - D(\boldsymbol{L},\boldsymbol{\gamma})\rho_\mu^2 J_i J_j \gamma_k\right), \quad \{L_i, \gamma_j\} = \varepsilon_{ijk}\gamma_k, \quad \{\gamma_i, \gamma_j\} = 0, \tag{25}$$

and the Hamiltonian is the energy (6), which can be written in the form:

$$H = \frac{\det \mathbf{J}}{2}\left(\left(1 - D\left(\boldsymbol{\gamma}, \mathbf{J}^{-1}\boldsymbol{\gamma}\right)\right)\left(\boldsymbol{L}, \mathbf{J}^{-1}\boldsymbol{L}\right) + D\left(\mathbf{J}^{-1}\boldsymbol{L}, \boldsymbol{\gamma}\right)^2\right) + U(\boldsymbol{\gamma}). \tag{26}$$

Now we show that the system (20) describing the motion of Chaplygin's ball is a generalized Chaplygin system (1), and the bracket (25) can be obtained using the method of reducing multiplier (see theorem 1).

As in the Veselova problem, we use now the local coordinates: the Euler angles θ, φ, ψ and the Cartesian coordinates of the ball's center x, y. In the moving frame of reference aligned with the ball's principal axes, the angular velocity vector and the normal to the plane are given by (10).

The equations of the constraints (corresponding to the no slip condition at the point of contact) can be written in the form

$$f_x = \dot{x} - R\dot{\theta}\sin\psi + R\dot{\varphi}\sin\theta\cos\psi = 0, \quad f_y = \dot{y} + R\dot{\theta}\cos\psi + R\dot{\varphi}\sin\theta\sin\psi = 0. \tag{27}$$

The equations of motion with Lagrange multipliers are follows:

$$\begin{gathered}
\frac{d}{dt}\left(\frac{\partial T_0}{\partial \dot{x}}\right) = \lambda_x, \quad \frac{d}{dt}\left(\frac{\partial T_0}{\partial \dot{y}}\right) = \lambda_y, \quad \frac{d}{dt}\left(\frac{\partial T_0}{\partial \dot{\psi}}\right) = 0, \\
\frac{d}{dt}\left(\frac{\partial T_0}{\partial \dot{\theta}}\right) - \frac{\partial T_0}{\partial \theta} = \lambda_x\frac{\partial f_x}{\partial \dot{\theta}} + \lambda_y\frac{\partial f_y}{\partial \dot{\theta}}, \quad \frac{d}{dt}\left(\frac{\partial T_0}{\partial \dot{\varphi}}\right) - \frac{\partial T_0}{\partial \varphi} = \lambda_x\frac{\partial f_x}{\partial \dot{\varphi}} + \lambda_y\frac{\partial f_y}{\partial \dot{\varphi}},
\end{gathered} \tag{28}$$

where T_0 is the ball's kinetic energy without taking into account the constraints (27) (obviously, this energy does not depend on x, y, and ψ):

$$T_0 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}(\boldsymbol{\omega}, \mathbf{I}\boldsymbol{\omega}).$$

Eliminating the undetermined multipliers λ_x and λ_y with the help of the first two equations in (28) and the constraints (27) we get

$$\begin{aligned}
\lambda_x\frac{\partial f_x}{\partial \dot{\theta}} + \lambda_y\frac{\partial f_y}{\partial \dot{\theta}} &= -mR^2(\ddot{\theta} + \dot{\psi}\dot{\varphi}\sin\theta), \\
\lambda_x\frac{\partial f_x}{\partial \dot{\varphi}} + \lambda_y\frac{\partial f_y}{\partial \dot{\varphi}} &= -mR^2(\ddot{\varphi}\sin\theta + \dot{\theta}\dot{\varphi}\cos\theta - \dot{\theta}\dot{\psi})\sin\theta.
\end{aligned}$$

Hence, the equations of motion for the angles θ and φ do not depend on ψ, but only on $\dot{\psi}$. Therefore, ψ is a cyclic variable and can be eliminated using the Routh reduction procedure;

after that the equations of motion for θ and φ can be written as

$$\begin{gathered}\frac{d}{dt}\left(\frac{\partial\mathcal{R}}{\partial\dot\theta}\right)-\frac{\partial\mathcal{R}}{\partial\theta}=-\dot\varphi S,\quad \frac{d}{dt}\left(\frac{\partial\mathcal{R}}{\partial\dot\varphi}\right)-\frac{\partial\mathcal{R}}{\partial\varphi}=\dot\theta S,\\ S=mR^2\sin\theta\left(\dot\varphi\cos\theta+\dot\psi\right).\end{gathered}\tag{29}$$

Here, $\mathcal{R}$ is the Routh function:

$$\mathcal{R}(\theta,\varphi,\dot\theta,\dot\varphi)=T_0-\dot\psi\frac{\partial T_0}{\partial\dot\psi},$$

from which $\dot x$ and $\dot y$ should be eliminated using the equations of constraints, while $\dot\psi$ is eliminated using the equation for the cyclic integral,

$$\begin{aligned}\frac{\partial T_0}{\partial\dot\psi}=(I_1-I_2)\dot\theta\sin\theta\sin\varphi\cos\varphi+I_3\dot\varphi\cos\theta+&\\ +\left((I_1\sin^2\varphi+I_2\cos^2\varphi)\sin^2\theta+I_3\cos^2\theta\right)\dot\psi=c=\text{const.}&\end{aligned}\tag{30}$$

So, we get the equations of motion in the form of a generalized Chaplygin system (1), and, since the system has an invariant measure, it is possible to write equations (29) in the Hamiltonian form with the bracket (3).

Perform the time substitution of the form

$$N(\theta,\varphi)dt=d\tau,\tag{31}$$

where $N=\rho_\mu$ is the density of the invariant measure (22).

According to (2), the equations of motion, in terms of the new time, are

$$\begin{gathered}\frac{d}{d\tau}\left(\frac{\partial\bar{\mathcal{R}}}{\partial\theta'}\right)-\frac{\partial\bar{\mathcal{R}}}{\partial\theta}=-\varphi'\bar S,\quad \frac{d}{d\tau}\left(\frac{\partial\bar{\mathcal{R}}}{\partial\varphi'}\right)-\frac{\partial\bar{\mathcal{R}}}{\partial\varphi}=\theta'\bar S,\\ \bar S=c(I_3+mR^2)mR^2N^3\sin\theta(I_1\cos^2\varphi+I_2\sin^2\varphi+mR^2),\end{gathered}\tag{32}$$

where $\theta'=\dfrac{d\theta}{d\tau}=N^{-1}\dot\theta$, $\varphi'=\dfrac{d\varphi}{d\tau}=N^{-1}\dot\varphi$, $\bar{\mathcal{R}}=\mathcal{R}(\theta,\dot\theta,\varphi,\dot\varphi)\big|_{\dot\theta=N\theta',\ \dot\varphi=N\varphi'}$.

By applying the Legendre transformation to the system (32) we arrive at

Theorem 4. *Upon the time substitution $\rho_\mu dt=d\tau$ the equations of motion* (29) *for Chaplygin's ball can be written in the Hamiltonian form*:

$$\frac{d\theta}{d\tau}=\frac{\partial H}{\partial p_\theta},\quad \frac{dp_\theta}{d\tau}=-\frac{\partial H}{\partial\theta}-\bar S\frac{\partial H}{\partial p_\varphi},\quad \frac{d\varphi}{d\tau}=\frac{\partial H}{\partial p_\varphi},\quad \frac{dp_\varphi}{d\tau}=-\frac{\partial H}{\partial\varphi}+\bar S\frac{\partial H}{\partial p_\theta},$$

with the Poisson bracket of the form

$$\{\theta,p_\theta\}=\{\varphi,p_\varphi\}=1,\quad \{p_\varphi,p_\theta\}=\bar S(\theta,\varphi),\quad \{\theta,\varphi\}=0,\tag{33}$$

where

$$
\begin{gathered}
p_\theta = \frac{\partial \bar{\mathcal{R}}}{\partial \theta'}, \quad p_\varphi = \frac{\partial \bar{\mathcal{R}}}{\partial \varphi'},\\
H = \theta' p_\theta + \varphi' p_\varphi - \bar{\mathcal{R}} =\\
= \frac{1}{2} p_\theta^2 \left(I_3 \widetilde{I}_{12} - D(\boldsymbol{\gamma}, \mathbf{I}\boldsymbol{\gamma}) \right) + + \frac{1}{2} \frac{p_\varphi^2}{\sin^2\theta} \left(I_1 I_2 \sin^2\theta + I_3 \widetilde{I}_{12} \cos^2\theta - D(\boldsymbol{\gamma}, \mathbf{I}\boldsymbol{\gamma}) \right) +\\
+ \frac{p_\theta p_\varphi}{\sin\theta} I_3 (I_1 - I_2) \cos\theta \sin\theta \sin\varphi \cos\varphi - \frac{N c p_\theta}{\sin\theta} (I_1 - I_2)(I_3 + D \sin^2\theta) \sin\varphi \cos\varphi -\\
- \frac{N c p_\varphi}{\sin^2\theta} I_3 (\widetilde{I}_{21} + D) + \frac{N^2 c^2}{\sin^2\theta} (I_3 + D \sin^2\theta)(\widetilde{I}_{21} + D),\\
\widetilde{I}_{12} = I_1 \sin^2\varphi + I_2 \cos^2\varphi, \quad \widetilde{I}_{21} = I_1 \cos^2\varphi + I_2 \sin^2\varphi.
\end{gathered}
$$

Expressing the variables $\boldsymbol{L} = \rho_\mu \boldsymbol{M}$ (24) in terms of the local variables θ, φ, p_θ, and p_φ, we find

$$
L_1 = p_\theta \cos\varphi - p_\varphi \frac{\cos\theta \sin\varphi}{\sin\theta} + cN \frac{\sin\varphi}{\sin\theta}, \quad L_2 = -p_\theta \sin\varphi - p_\varphi \frac{\cos\theta \cos\varphi}{\sin\theta} + cN \frac{\cos\varphi}{\sin\theta}, \quad L_3 = p_\varphi.
$$

Using such a transformation one can straightforwardly obtain (25).

Let us consider in more detail the integrable case $U = 0$ with zero constant of areas $(\boldsymbol{M}, \boldsymbol{\gamma}) = (\boldsymbol{L}, \boldsymbol{\gamma}) = 0$, because the bracket (25) in this case corresponds to the algebra $e(3)$. We write the Hamiltonian (21) (omitting unessential multipliers) and the additional integral (23) in terms of the variables $\boldsymbol{L}$, $\boldsymbol{\gamma}$ as follows

$$
\begin{gathered}
H = \frac{1}{2} \boldsymbol{L}^2 (\boldsymbol{\gamma}, \mathbf{B}\boldsymbol{\gamma}) - \frac{1}{2} \left[(\boldsymbol{L}, \mathbf{B}\boldsymbol{L})(\boldsymbol{\gamma}, \mathbf{B}\boldsymbol{\gamma}) - (\boldsymbol{\gamma}, \mathbf{B}\boldsymbol{L})^2 \right],\\
F = \boldsymbol{L}^2 (\boldsymbol{\gamma}, \mathbf{B}\boldsymbol{\gamma}), \quad \mathbf{B} = 1 - D\mathbf{J}^{-1} = \mathbf{I}\mathbf{J}^{-1}.
\end{gathered}
$$

Using proposition 1 from the previous section, we get the following result:

Theorem 5. *On a fixed level* $(\boldsymbol{M}, \boldsymbol{M}) = \mathrm{const}$ *and* $(\boldsymbol{M}, \boldsymbol{\gamma}) = 0$ *the vector field* (20), *with* $U = 0$, *after the time substitution* (24) *and the change of variables*

$$
\boldsymbol{s} = (\boldsymbol{\gamma}, \mathbf{B}\boldsymbol{\gamma})^{-1/2} \mathbf{B}^{-1/2} \boldsymbol{\gamma}, \quad \widetilde{\boldsymbol{L}} = \mathbf{B}^{-1/2} \boldsymbol{L}
$$

is reduced to the vector field of the Clebsch case in the Kirchhoff equations with zero constant of areas.

4. Realization of constraints. Chaplygin's ball with the Veselova constraint

In the paper by A. P. Veselov and L. Ye. Veselova [7], as well as in [17], the authors consider the problem of rolling on a plane of a balanced, dynamically asymmetric ball (Chaplygin's ball)

with an additional nonholonomic constraint (the Veselova constraint):

$$(\boldsymbol{\omega}, \boldsymbol{E}) = 0,$$

where $\boldsymbol{E}$ is the unit vector of a space-fixed axis.

The case $\boldsymbol{E} \perp \boldsymbol{\gamma}$, with $\boldsymbol{\gamma}$ being normal to the plane of contact, was considered in [7] where the system was proved to be integrable according to the Euler–Jacobi theorem. Besides, a realization of this constraint by means of absolutely smooth walls was offered in [7] (see Fig. 1) so that the ball moves along the straight line on the plane.

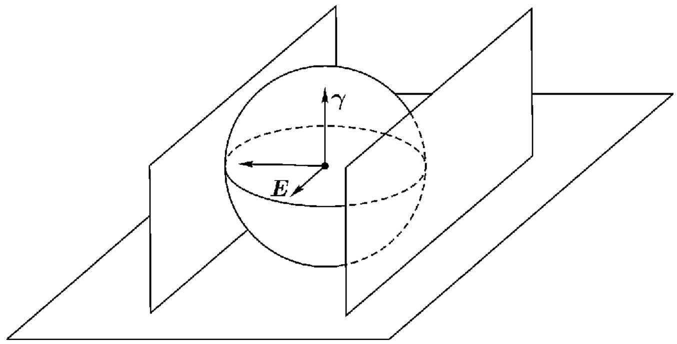

Fig 1: A ball rolling along a straight line (A. P. Veselov, L. Ye. Veselova).

In [17], the case $\boldsymbol{E} \parallel \boldsymbol{\gamma}$ was considered and it was supposed that this system should describe the motion of a (massive) rubber ball on a plane (an ordinary Chaplygin's ball, considered above, was called in this paper a marble ball). Later, we will show that this system is equivalent to the Veselova system (5) (and thereby answer the question of the integrability of a rubber ball's motion on a plane, formulated in [17]).

Here, we consider the general situation, assuming that $\boldsymbol{E}$ is an arbitrary space-fixed unit vector. We start with describing a possible realization of such a system (or constraints), since it is obvious that the ball-upon-a-plane realization in this case is impossible. Basing on results of [20, 18, 14, 16], we offer a more general realization for compositions of Chaplygin constraints, Suslov constraints and Veselova constraints, using only perfect rolling (without dissipation of energy).

Let us start with a *spherical support* [20]. This system describes the motion of a rigid body with a fixed point O which is enclosed in a spherical shell; the shell touches an arbitrary number of massive dynamically symmetric balls with fixed centers (Fig. 2). It is supposed that there is no sliding at the points of contact of the balls and the shell. As was shown in [20] this system is integrable with an arbitrary number of balls. In the case of a single external ball, we have a problem equivalent to the problem of Chaplygin's ball.

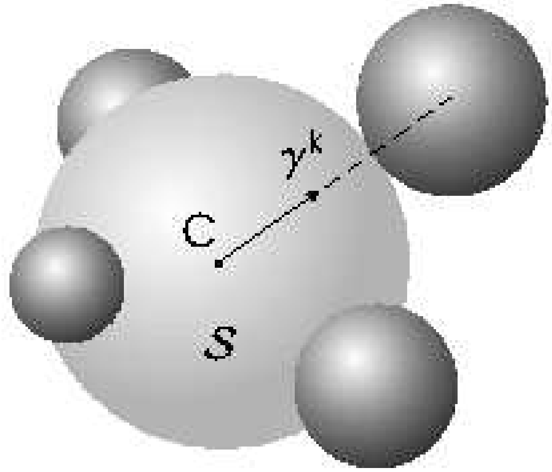

Fig 2: The spherical support (Yu. N. Fedorov)

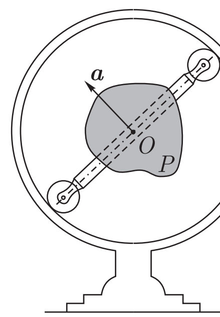

Fig 3: Wagner's realization of the Suslov constraint

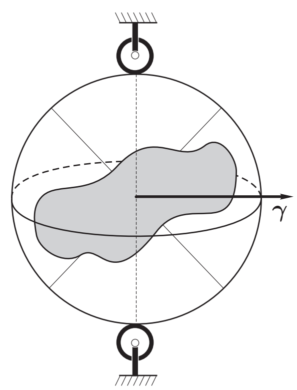

Fig 4: Realization of the Veselova constraint

Another system, which can be used for realization of nonholonomic constraints, is called a *nonholonomic joint* [14]. The original version of this construction was offered in [18] as a realization of the constraint of the Suslov problem $(\boldsymbol{\omega}, \boldsymbol{a}) = 0$, where $\boldsymbol{a}$ is a body-fixed vector. In this case, flat wheels (disks) are attached to the body with the fixed point; these wheels roll without sliding upon the interior surface of a fixed spherical shell (see Fig. 3). It is supposed

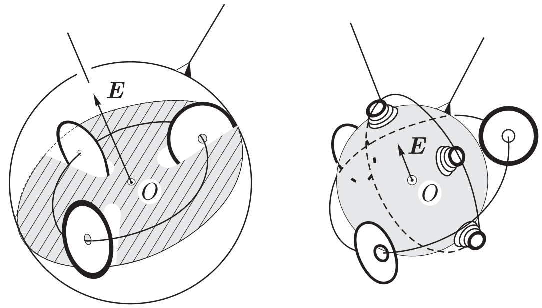

Fig 5: The nonholonomic joint

that a wheel is so sharp that its velocity in the direction perpendicular to its plane is zero. Similarly, one can consider the motion of a body with a fixed point when the body is enclosed in a spherical shell, which is touched by wheels (disks), whose axes are fixed in space (Fig. 4). It is clear that in the simplest case, we obtain the constraint of the Veselova problem $(\boldsymbol{\omega}, \boldsymbol{\gamma}) = 0$, where $\boldsymbol{\gamma}$ is the vector lying in the disk's plane (this result was mentioned in [16]). In [14], a similar realization of a similar constraint was offered: a frame with wheels (disks) is attached to the interior or exterior surface of a spherical shell (Fig. 5). Such a construction ensures equality of projections of the angular velocities $\boldsymbol{\omega}_s$ of the spherical shell and $\boldsymbol{\omega}_f$ of the frame with wheels (and, correspondingly, of the bodies attached to them) onto the axis $\boldsymbol{E}$ which is orthogonal to the plane containing the disks' axes:

$$(\boldsymbol{\omega}_s, \boldsymbol{E}) = (\boldsymbol{\omega}_f, \boldsymbol{E}).$$

If the frame with wheels is fixed in space, then we have the Veselova constraint, while the space-fixed spherical shell gives the Suslov constraint.

Now let us consider a combination of a spherical support and a nonholonomic joint, where the body with a fixed point is enclosed in a spherical shell which is in contact with a single ball and a single disk (see Fig. 6).

In the frame of reference aligned with the principal axes of the body, the equations of constraints read

$$R\boldsymbol{\omega} \times \boldsymbol{\gamma} + R_1\boldsymbol{\omega}_1 \times \boldsymbol{\gamma} = 0, \quad (\boldsymbol{\omega}, \boldsymbol{E}) = 0, \tag{34}$$

where $\boldsymbol{\omega}$ is the body's angular velocity, R is the radius of the spherical shell, $\boldsymbol{\omega}_1$ and R_1 are the angular velocity and radius of the adjoining ball, $\boldsymbol{\gamma}$ is the unit vector of the axis through the balls' centers, and $\boldsymbol{E}$ is the normal vector to the plane that contains the ball's center and the disk's axis.

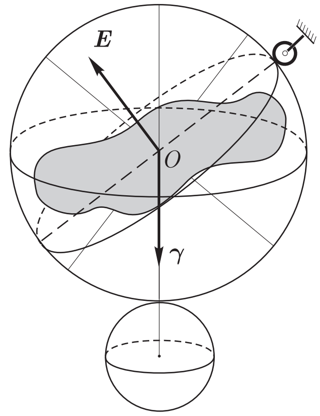

Fig 6:

The equations of motion with undetermined multipliers look as follows:

$$\begin{gathered}\mathbf{I}\dot{\boldsymbol{\omega}} = \mathbf{I}\boldsymbol{\omega} \times \boldsymbol{\omega} + R\boldsymbol{\gamma} \times \boldsymbol{N} + \mu\boldsymbol{E} + \boldsymbol{M}_Q, \quad D_1\dot{\boldsymbol{\omega}}_1 = D_1\boldsymbol{\omega}_1 \times \boldsymbol{\omega} + R_1\boldsymbol{\gamma} \times \boldsymbol{N}, \\ \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\omega}, \quad \dot{\boldsymbol{E}} = \boldsymbol{E} \times \boldsymbol{\omega},\end{gathered} \tag{35}$$

where $\mathbf{I} = \operatorname{diag}(I_1, I_2, I_3)$ is the body's tensor of inertia, D_1 is the scalar tensor of inertia of the adjoining ball, $\boldsymbol{N} = (N_1, N_2, N_3)$ and μ are the undetermined multipliers that correspond to reaction of the constraints (34), and $\boldsymbol{M}_Q$ is the moment of the external forces. Using the second equation of (35) we find that $(\boldsymbol{\omega}_1, \boldsymbol{\gamma})^{\boldsymbol{\cdot}} = 0$, therefore,

$$(\dot{\boldsymbol{\omega}}_1, \boldsymbol{\gamma}) = -(\boldsymbol{\omega}_1, \dot{\boldsymbol{\gamma}}) = -(\boldsymbol{\omega}_1, \boldsymbol{\gamma} \times \boldsymbol{\omega}).$$

Using this relation and the second equation in (35), we eliminate $\boldsymbol{\gamma} \times \boldsymbol{N}$ from the remaining equations. As a result, we obtain

$$\mathbf{I}\dot{\boldsymbol{\omega}} + D\boldsymbol{\gamma} \times (\dot{\boldsymbol{\omega}} \times \boldsymbol{\gamma}) = \mathbf{I}\boldsymbol{\omega} \times \boldsymbol{\omega} + \mu\boldsymbol{E} + \boldsymbol{M}_Q, \quad D = \frac{R^2}{R_1^2}D_1, \quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\omega}, \quad \dot{\boldsymbol{E}} = \boldsymbol{E} \times \boldsymbol{\omega}. \tag{36}$$

Comparing this with (5) and (20), we conclude that these equations coincide with the equations for the rolling of Chaplygin's ball with the additional Veselova constraint; in this case the direction of the vector $\boldsymbol{E}$ can be arbitrary. The undetermined multiplier μ can be found from the relation $(\boldsymbol{E}, \boldsymbol{\omega})^{\boldsymbol{\cdot}} = 0$:

$$\mu = -\frac{(\mathbf{I}\boldsymbol{\omega} \times \boldsymbol{\omega} + \boldsymbol{M}_Q, \mathbf{I}^{-1}\boldsymbol{E})}{(\boldsymbol{E}, \mathbf{I}_Q^{-1}\boldsymbol{E})}, \quad \mathbf{I}_Q = \mathbf{J} - D\boldsymbol{\gamma} \otimes \boldsymbol{\gamma}, \quad \mathbf{J} = \mathbf{I} + D.$$

By straightforward calculations one can show that if $\boldsymbol{M}_Q$ does not depend on $\boldsymbol{\omega}$, equations (36) have the invariant measure $\rho_\omega\, d^3\boldsymbol{\omega}\, d^3\boldsymbol{\gamma}$ with density

$$\rho_\omega = \left((\boldsymbol{E}, \mathbf{I}_Q^{-1}\boldsymbol{E}) \det \mathbf{I}_Q\right)^{1/2}. \tag{37}$$

There are also obvious geometric integrals

$$\boldsymbol{\gamma}^2 = 1, \quad \boldsymbol{E}^2 = 1, \quad (\boldsymbol{\gamma}, \boldsymbol{E}) = \text{const}.$$

In the potential force field $\boldsymbol{M}_Q = \boldsymbol{\gamma} \times \dfrac{\partial U}{\partial \boldsymbol{\gamma}} + \boldsymbol{E} \times \dfrac{\partial U}{\partial \boldsymbol{E}}$ and the energy is also conserved:

$$H = \frac{1}{2}(\mathbf{I}_Q \boldsymbol{\omega}, \boldsymbol{\omega}) + U(\boldsymbol{\gamma}, \boldsymbol{E}),$$

where $U(\boldsymbol{\gamma}, \boldsymbol{E})$ is the potential energy of the external forces. If there are no external forces ($U = 0$) and ($\boldsymbol{E} \times \boldsymbol{\gamma} \neq 0$), then there are two additional integrals:

$$\begin{gathered} F_1 = (\boldsymbol{K}, \boldsymbol{E} \times \boldsymbol{\gamma}), \quad F_2 = (\boldsymbol{K}, \boldsymbol{E} \times (\boldsymbol{E} \times \boldsymbol{\gamma})), \\ \boldsymbol{K} = \mathbf{I}_Q \boldsymbol{\omega} - (\mathbf{I}_Q \boldsymbol{\omega}, \boldsymbol{E})\boldsymbol{E}; \end{gathered} \tag{38}$$

hence, the system (36) is integrable (according to the Euler–Jacobi theorem).

In order to prove that the integrals F_1, F_2 exist, let us write the equations of evolution of the vector $\boldsymbol{K}$:

$$\dot{\boldsymbol{K}} = \boldsymbol{K} \times \boldsymbol{\omega}.$$

Hence, the vector $\boldsymbol{K}$ is fixed in space, and all its projections onto fixed axes are conserved, but since $(\boldsymbol{K}, \boldsymbol{E}) \equiv 0$, only two independent integrals remain.

Therefore, the system (36) (when $\boldsymbol{E} \times \boldsymbol{\gamma} \neq 0$) is almost identical to the system considered in [7] (where an additional constraint $(\boldsymbol{E}, \boldsymbol{\gamma}) = 0$ is imposed). Note that the system (36) with $U = 0$ has not yet been integrated in terms of quadratures.

In the special case $\boldsymbol{E} = \boldsymbol{\gamma}$ the integrals (38) are identically zero, but from the equation of the constraint $(\boldsymbol{\omega}, \boldsymbol{\gamma}) = 0$ we find that $(\boldsymbol{\omega}, \boldsymbol{\gamma})^{\bullet} = (\dot{\boldsymbol{\omega}}, \boldsymbol{\gamma}) = 0$ and, hence,

$$\mathbf{I}_Q \boldsymbol{\omega} = \mathbf{J}\boldsymbol{\omega} - D(\boldsymbol{\omega}, \boldsymbol{\gamma})\boldsymbol{\gamma} = \mathbf{J}\boldsymbol{\omega}, \quad \mathbf{I}_Q \dot{\boldsymbol{\omega}} = \mathbf{J}\boldsymbol{\omega} - D(\dot{\boldsymbol{\omega}}, \boldsymbol{\gamma})\boldsymbol{\gamma} = \mathbf{J}\dot{\boldsymbol{\omega}}, \quad \mathbf{I}\boldsymbol{\omega} \times \boldsymbol{\omega} = \mathbf{J}\boldsymbol{\omega} \times \boldsymbol{\omega}.$$

Therefore, in this case after the change $\mathbf{I} \to \mathbf{J}$ the system (36) is equivalent to the Veselova system (5). Thus, we *answer the question of integrability of rolling of a rubber ball formulated in [17].*

5. Nonholonomic Jacobi problem

Here we consider another class of nonholonomic systems with an invariant measure and first integrals; however, it is still unknown, if the systems' equations of motion can be written in the Hamiltonian form.

Let a dynamically symmetric ball ($\mathbf{I} = D_0\mathbf{E}$) of radius R roll without sliding upon a fixed surface. In a fixed Cartesian frame of reference, the surface, on which the ball's center moves, is given by $\Phi(\boldsymbol{x}) = 0$. (It is clear that the surface, on which the ball moves, is equidistant to the surface $\Phi(\boldsymbol{x}) = 0$, but the equations of motion take a simpler form if the radius-vector of the center $\boldsymbol{x}$ is used.)

The equations of the constraint and the equations of motion with undetermined multipliers are

$$\boldsymbol{v} - R\boldsymbol{\omega} \times \boldsymbol{n} = 0, \quad \boldsymbol{n} = \frac{\nabla\Phi(\boldsymbol{x})}{|\nabla\Phi(\boldsymbol{x})|},$$
$$m\dot{\boldsymbol{v}} = \boldsymbol{N} - \frac{\partial U}{\partial \boldsymbol{x}}, \quad \mu\dot{\boldsymbol{\omega}} = R\boldsymbol{N} \times \boldsymbol{n},$$

where $\boldsymbol{v} = \dot{\boldsymbol{x}}$ is the velocity of the center of mass, $\boldsymbol{\omega}$ is the ball's angular velocity, m is the mass of the ball, $\boldsymbol{N}$ is the constraint reaction, $\boldsymbol{n}$ is the normal to the surface, and $U(\boldsymbol{x})$ is the potential energy of the external field. All the vectors are supposed to be projected onto the fixed system.

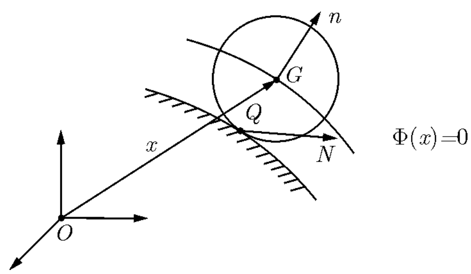

Fig 7: Rolling of a ball on a surface (G is the center of mass, Q is the point of contact of the ball and the surface)

After eliminating the undetermined multipliers $\boldsymbol{N}$, we obtain the equations of motion in the form

$$\left(m + \frac{D_0}{R^2}\right)\dot{\boldsymbol{\omega}} = m(\boldsymbol{\omega}, \boldsymbol{n})\dot{\boldsymbol{n}} + R\boldsymbol{n} \times \frac{\partial U}{\partial \boldsymbol{x}}, \quad \dot{\boldsymbol{x}} = R\boldsymbol{\omega} \times \boldsymbol{n}, \tag{39}$$

where $\dot{\boldsymbol{n}}$ is expressed, using the equation of the surface, as follows:

$$\dot{n}_i = \frac{1}{|\nabla\Phi|}\sum_k \left(\frac{\partial^2\Phi}{\partial x_i \partial x_k} - \frac{1}{|\nabla\Phi|^2}\sum_j \left(\frac{\partial\Phi}{\partial x_i}\right)\left(\frac{\partial\Phi}{\partial x_j}\right)\frac{\partial^2\Phi}{\partial x_j \partial x_k}\right)\dot{x}_k.$$

Equations (39) admit the geometric integral and the integral of energy

$$\Phi(\boldsymbol{x}) = 0, \quad H = \frac{1}{2}(D_0 + D)\boldsymbol{\omega}^2 - \frac{1}{2}\mu(\boldsymbol{\omega}, \boldsymbol{n})^2 + U(\boldsymbol{x}), \quad D = mR^2. \tag{40}$$

Besides, there is also the invariant measure $\rho_\omega\, d^2\boldsymbol{\omega}\, d^3\boldsymbol{x}$ with density [26]

$$\rho_\omega = |\nabla\Phi(\boldsymbol{x})| = \sqrt{\sum_{k=1}^{3}\left(\frac{\partial\Phi}{\partial x_k}\right)^2}. \tag{41}$$

Is it possible (using, say, the method of reducing multiplier) to represent the equations (39) *in the Hamiltonian form*?

Obviously, in the general case the answer to this question is 'no'. Indeed, writing (39) in terms of the local coordinates on the surface $\Phi(\boldsymbol{x}) = 0$, we obtain the system of five equations with the integral of energy. If this system is Hamiltonian, then the corresponding Poisson bracket is necessarily degenerate (the rank of the Poisson bracket is always even); consequently, there should exist a Casimir function $F(\boldsymbol{x}, \boldsymbol{\omega})$ independent of the energy (40). But if such a function existed, it would have been an integral of equations (39), which, as numerical experiments show, generally does not exist.

There is an important special case of (39), when the surface is a three-axial ellipsoid $\Phi(\boldsymbol{x}) = (\boldsymbol{x}, \mathbf{B}^{-1}\boldsymbol{x}) - 1 = 0$, $\mathbf{B} = \operatorname{diag}(b_1, b_2, b_3)$ (more precisely, an arbitrary quadric surface). This is the so-called *nonholonomic Jacobi problem* [4], similar to the problem of geodesics on an ellipsoid. It is shown in [4] that in this case there is an additional integral

$$K = \frac{(\boldsymbol{\omega} \times \boldsymbol{n}, \mathbf{B}^{-1}\boldsymbol{\omega} \times \boldsymbol{n})}{(\boldsymbol{n}, \mathbf{B}\boldsymbol{n})}. \tag{42}$$

The integral (42) is similar to the Joachimstahl integral in the classical problem of geodesics and can be extended to the case when the potential $U(x) = \frac{k}{2}\boldsymbol{x}^2 + \frac{1}{2}\sum_i \frac{c_i}{x_i^2}$, $k, c_i = \text{const}$, is added.

If the nonholonomic Jacobi problem can be written in the Hamiltonian form (possibly with time substitution)? It is a very complicated question. On one hand, as numerical experiments show, the system in this case behaves chaotically and is nonintegrable [4], so we cannot find here any obstacles for Hamiltonian representation, typical for integrable systems. Besides, the two-dimensional Poincaré map, which can be constructed on the level surface of the integrals (40) and (42), has a measure and is symplectic; therefore, this map can be generated into the flow of a Hamiltonian system (see, for example, [27]). On the other hand, the method of reducing multiplier and the explicit Poisson structure fitting do not give the required results. Note that the possibility of hamiltonization essentially depends on smoothness, analiticity, or algebraicity of the sought-for Poisson structure. Here, we do not consider these issues.

Another nonholonomic problem (somewhat simpler than the previous one, because there exists a certain integrable limit problem) concerns the system considered in [21]: it is a problem of the *spherical suspension*. It is supposed in this case that a dynamically asymmetric ball (Chaplygin's ball) rolls on the surface of a fixed sphere. An analysis of the problem's integrability is given in [21, 29].

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